(8 Pages) **Reg. No. : ....**

Code No. : 6304 **Sub. Code : PMAM 12/ ZMAM 12**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2021

First Semester

Mathematics — Core

ANALYSIS — I

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answers :

1. Let (X,d) be a metric space when X is the set of all real numbers. If every infinite subset of X is open then

   (a) all finite sets are not open

   (b) all finite sets are also open

   (c) d is usual metric

   (d) can not say

2. Every interval $[a,b]$ $a<b$ is

(a) countable and perfect

(b) uncountable and not perfect

(c) uncountable and perfect

(d) countable and not perfect

3. If $p>0$, then $\lim_{n\to\infty}\left(\frac{1}{p}\right)^{\frac{1}{n}}=$

(a) 0 (b) 1

(c) ∞ (d) not convergent

4. If $|x|>1$, then $\lim_{n\to\infty} x^n =$

(a) 0 (b) 1

(c) ∞ (d) ∞ or $-\infty$

5. Let $a_n \geq 0 \forall n$ and $\alpha = \lim_{n\to\infty}\sup\left(\frac{a_{n+1}}{a_n}\right)$ Then , $\sum_n a_n$ converges if α

(a) >1 (b) =1

(c) <1 (d) >0

6. The radius of convergence of the series $\sum_n \frac{z^n}{n^3}$ is

(a) 0 (b) 1

(c) ∞ (d) 3

7. The function $f(x)=\begin{cases}\sqrt{2} & x \text{ is rational} \\ x & \text{otherwise}\end{cases}$ is discontinuous at

(a) of first kind at $\sqrt{2}$

(b) of second kind at $\sqrt{2}$

(c) of first kind at $\mathbb{R}-(\sqrt{2})$

(d) of second kind at $\mathbb{R}-(\sqrt{2})$

8. Let $f : X \to Y$ be a monotonic decreasing function. Then the number of discontinuities of first kind is

(a) 0

(b) has to be finite

(c) atmost countably infinite

(d) can be uncountably infinite

9. Let f be defined for all real numbers and suppose that for all real numbers x, y $|f(x)-f(y)| \le (x-y)^2$, then

(a) f is monotonically increasing

(b) f is monotonically decreasing

(c) f is constant

(d) none of the above

10. Let f be a differentiate function. Then the number of simple discontinuities of f' is

(a) 0

(b) atmost countably infinity

(c) can be uncountably infinite

(d) none of the above

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b)

11. (a) Define the terms neighborhood, limit point. Show that if p is a limit point of a set E, then every neighborbood of p contains infinitely many points of E.

Or

(b) Define connected set. What are the connected subsets of the real line. Justify your answer.

12. (a) Define terms subsequence, subsequential limit. Prove that the subsequential limits of a sequence (p_n) in a metric space X is a closed set in X.

Or

(b) Define the terms monotonic sequence, bounded sequence. Prove that a bounded monotonic sequence is convergent.

13. (a) State and prove ratio test.

Or

(b) State and prove Leibnitz theorem.

14. (a) (i) Let X, Y be two metric spaces. Let $f : X \to Y$ be a continuous mapping. Then prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every subset E of X.

(ii) Also prove that this inclusion can be proper.

Or

(b) (i) Define discontinuities of first kind and second kind.

(ii) Prove that if f is a monotonic function defined on (a,b), the number of points at which f is discontinuous is not uncountable.

15. (a) (i) Define local maximum.

(ii) If f is defined in $[a,b]$ and f has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exists then prove that it is 0. Also state L'Hospital's rule.

Or

(b) State and prove Taylor's theorem.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b)

16. (a) Define perfect set, Cantor set. Prove that Cantor set is perfect.

Or

(b) Define the terms compact set, k-cell. Prove that every k-cell is compact.

17. (a) (i) Define the number e.

(ii) Prove that $2 < e < 3$.

(iii) Prove that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

(iv) Prove that the number e is not rational.

Or

(b) (i) Define the terms convergent sequence and Cauchy sequence.

(ii) Prove that convergence sequence is a Cauchy sequence and the converse holds if the space is compact.

(iii) Prove that in $\mathbb{R}^k$, a Cauchy sequence is convergent.

18. (a) (i) Define absolute convergence of a series.

(ii) Prove that absolute convergence implies convergence but not conversely.

(iii) If $\sum_n a_n = A; \sum_n b_n = B; \sum_n a_n$ converges absolutely and $c_n = \sum_{k=0}^{n} a_n b_{n-k}$, then prove that $\sum_n c_n = AB$.

Or

(b) State and prove root test. Deduce that $\frac{2}{3} + \frac{3}{5} + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{5}\right)^3 + \cdots$ is convergent.

19. (a) Define compact set and connected set. Prove that the continuous image of a compact is compact and connected set is connected.

Or

(b) (i) Define monotonic functions.

(ii) Let f be monotonically increasing on (a,b). Prove that $f(x,+)$ and $f(x,-)$ exist for all x in (a,b) and is such that $\sup_{a<t<x} f(t) = f(x,-) \leq f(x) \leq f(x,+) = \inf_{x<t<b} f(t)$. Also prove that $a = x \leq y = b \Rightarrow f(x,+) \leq f(y,-)$

20. (a) (i) State and prove chain rule of differentiation.

(ii) Let f be defined as $f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$. Prove that $f'(0)$ does not exist.

(iii) Let f be defined as $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$. Prove that $f'(0) = 0$.

Or

(b) (i) State and prove Generalised mean value theorem.

(ii) Discuss the behaviour of f according as (1) $f'(x) \geq 0$; (2) $f'(x) = 0$; (3) $f'(x) \leq 0$.

(iii) Suppose f is real differentiable on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$, then there exists x such that $a < x < b$ and $f'(x) = \lambda$.